M4P52 Manifolds, 2016
Problem Sheet 1

1. Let X and Y be n-dimensional topological manifolds. Prove that the disjoint union $X \sqcup Y$ is an n-dimensional topological manifold.

   Is $S^1 \sqcup S^2$ a topological manifold?

2. Recall that that the *discrete topology* on a set X is given by declaring that every 1-point set $\{x\} \subset X$ is open (and hence every subset of X is open). Prove that a topological space X is a topological manifold of dimension zero iff X has the discrete topology.

$\star$3$\star$. Let $X = S^1$, and let (U_1, f_1) and (U_2, f_2) be the two 'stereographic projection' co-ordinate charts described in Example 2.4. Define a new co-ordinate chart (U_0, f_0) by

$$U_0 = S^1 \cap \{y < 0\}, \qquad \tilde{U}_0 = (-1, 1) \subset \mathbb{R}$$

$$\begin{aligned} f_0 : U_0 &\xrightarrow{\sim} \tilde{U}_0 \\ (x, y) &\mapsto x \end{aligned}$$

   (a) Write down the transition functions $\phi_{10}, \phi_{01}, \phi_{20}$ and ϕ_{02}, including their domains and codomains.

   (b) Is (U_0, f_0) compatible with the atlas $\{(U_1, f_1), (U_2, f_2)\}$?

4. Prove Lemma 2.15.

5. Define an equivalence relation on $\mathbb{R}^2$ by

$$(x, y) \sim (x + n, (-1)^n y + m), \ \forall n, m \in \mathbb{Z}$$

   (these are the orbits of a group action generated by a vertical translation and a horizontal glide reflection). Let K be the topological space:

$$K = \mathbb{R}^2 / \sim$$

   (a) Find a smooth atlas on K. *Hint: copy Example 2.12.*

   (b) What does K look like?

6. Prove Corollary 2.20.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f : x \mapsto x^3$. Is $(\mathbb{R}, f)$ a co-ordinate chart on $\mathbb{R}$? Is it compatible with the standard smooth structure on $\mathbb{R}$?

8. Let $q : \mathbb{R}^2 \to T^2$ be the quotient map from Example 2.12. Let $\tilde{U} \subset \mathbb{R}^2$ be an open set such that $q|_{\tilde{U}}$ is injective, and let $U = q(\tilde{U})$.

   (a) Prove that $(q|_{\tilde{U}})^{-1} : U \to \tilde{U}$ is a co-ordinate chart on T^2.

   (b) Prove that this chart is compatible with the smooth structure from Example 2.12.

M4P52 Manifolds, 2016
Problem Sheet 2

1. Complete the proof of Proposition 2.26 by proving the two statements asserted in the final sentence.

2. Let X be any manifold, and let x be a point in X. Show that the subset $Z = \{x\}$ is a submanifold of X.

3. Let X be the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let $Z \subset X$ be the subset:

$$Z = \left\{[(x,y)] \in X \ ; \ y = \tfrac{1}{3}\sin(2\pi x) + m, \text{ for some } m \in \mathbb{Z}\right\}$$

Show that Z is a submanifold of X.

⋆3⋆. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and let

$$Z = \{(x, g(x)) \ ; \ x \in \mathbb{R}^n\} \ \subset \mathbb{R}^{n+1}$$

be the graph of g. Prove that Z is a submanifold of $\mathbb{R}^{n+1}$,

(a) without using Proposition 3.16,

(b) using Proposition 3.16.

4. Find a function $h : \mathbb{R}^n \to \mathbb{R}^k$ having a critical value $\alpha \in \mathbb{R}^k$ such that the level set $h^{-1}(\alpha) \subset \mathbb{R}^k$ is a submanifold of codimension k.

5. Let $Z = h^{-1}(1) \subset \mathbb{R}^3$ be the torus from Example 3.22. Following the procedure from that example and Example 3.21, construct a chart on Z containing the point $(3, 0, 0)$.

6. Show that the group

$$SL_2(\mathbb{R}) = \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \ ; \ \det(M) = 1\}$$

is a 3-dimensional submanifold of $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$. Now construct a chart on $SL_2(\mathbb{R})$ containing the identity matrix.

7. *(Advanced.)* Generalize Example 3.8 to show that, for any $k \leq m \leq n$, the Grassmannian $\mathrm{Gr}(k, m)$ sits as a submanifold inside $\mathrm{Gr}(k, n)$.

M4P52 Manifolds, 2016
Problem Sheet 3

1. (a) Let $F = (f, g) : \mathbb{R}^n \to \mathbb{R}^2$ be a smooth function, let α be a regular value of f, and let (α, β) be a regular value of F. Let $Y = f^{-1}(\alpha)$ and let $Z = F^{-1}(\alpha, \beta)$. Prove that Z is a submanifold of Y.

   Now let F be the function:

$$\begin{aligned} F : \mathbb{R}^4 &\to \mathbb{R}^2 \\ (x, y, z, w) &\mapsto (x^2 + y^2,\ z^2 + w^2) \end{aligned}$$

   (b) For any $\alpha \in (0, 1)$, show that the set $F^{-1}(\alpha, 1-\alpha)$ is a 2-dimensional submanifold of S^3.

   (c) What do these submanifolds look like? What happens at $\alpha = 0$ or 1?

2. Let X be the manifold $(\mathbb{R}, [\mathcal{C}])$ where $[\mathcal{C}]$ is the non-standard smooth structure from Example 2.25. Describe all the smooth functions from X to $\mathbb{R}$.

3. Show that a smooth function between two manifolds must be continuous.

⋆4⋆. Consider the function:

$$\begin{aligned} F : S^n &\to \mathbb{RP}^n \\ (x_0, ..., x_n) &\mapsto x_0 : ... : x_n \end{aligned}$$

   (a) Prove that F is smooth. *(Hint: use the charts on S^n from Example 3.21.)*

   (b) Is F surjective? Is it injective?

5. Fix $n \in \mathbb{Z}$, and consider the function:

$$\begin{aligned} F : T^1 &\to T^1 \\ [t] &\mapsto [nt] \end{aligned}$$

   Prove that F is smooth. Describe the level sets of F.

6. (a) Prove Lemma 4.7.

   (b) Prove Lemma 4.8.

M4P52 Manifolds, 2016
Problem Sheet 4

1. Prove Lemma 4.18.

2. (a) Prove Lemma 4.20.

   (b) Formulate and prove a 'dual' statement involving submersions.

3. Suppose $F : X \to Y$ is a submersion between a manifold of dimension n and a manifold of dimension k.

   (a) Show that at any point we can find co-ordinate charts which make F look like the standard projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$.

   (b) Deduce that the image of F is an open set in Y.

$\star 4 \star$. (a) Compute the rank of the function

$$\begin{aligned} F :\ & T^2 \to \mathbb{R}^3 \\ & [(s,t)] \mapsto \big(\cos 2\pi s(2+\cos 2\pi t),\ \sin 2\pi s(2+\cos 2\pi t),\ \sin 2\pi t\big) \end{aligned}$$

   at all points in T^2. *Hint: first consider points where* $\cos 2\pi t \neq 0$.

   (b) Consider the level set $Z_1 = h^{-1}(1) \subset \mathbb{R}^3$ of the function h from Example 3.20. Show that F defines a smooth function from T^2 to Z_1. Assuming that this a bijection, prove that it is a diffeomorphism.

5. Let X by the manifold $\mathbb{R}$ equipped with the standard smooth structure, and let Y be the manifold $\mathbb{R}$ equipped with the non-standard smooth structure $[\mathcal{C}]$ from Example 2.25. Prove that X and Y are diffeomorphic.

6. Prove that $\mathbb{RP}^1$ is diffeomorphic to T^1.

M4P52 Manifolds, 2016
Problem Sheet 5

⋆1⋆. Let $X = S^2$, and let $(U_1, f_1), (U_2, f_2)$ be the two stereographic projection charts from Example 2.5. Recall that the transition function between these two charts is

$$\phi_{21} : (x, y) \mapsto \frac{1}{r^2}(x, y)$$

where $r^2 = x^2 + y^2$ (see Example 2.7).

(a) Write down a curve σ in X through the point $(1, 0, 0)$, such that $\Delta_{f_1} : [\sigma] \mapsto (1, 0)$.

(b) Compute the Jacobian matrix of ϕ_{21} at the point $(1, 0)$, and then use this to find $\Delta_{f_2}([\sigma])$.

(c) If we view $T_{(1,0,0)}X$ as a subspace of $\mathbb{R}^3$ then what vector does $[\sigma]$ correspond to?

2. Let X, Y and Z be three manifolds, and let $F : X \to Y$ and $G : Y \to Z$ be smooth functions. Fix a point $x \in X$. Prove that the chain rule holds, *i.e.* that

$$D(G \circ F)|_x = DG|_{F(x)} \circ DF|_x$$

(a) by picking co-ordinate charts.

(b) without picking co-ordinate charts.

3. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the function $h(x, y) = xy$, and for any $\alpha \in \mathbb{R}$ let Z_α denote the corresponding level set of h.

(a) For $\alpha \neq 0$, find the tangent space to any point in Z_α as a subspace of $\mathbb{R}^2$.

(b) For any point $(x, y) \in Z_0$ find the kernel of $Dh|_{(x,y)}$.

4. Let X be a manifold of dimension n, and let $h, g : X \to \mathbb{R}$ be two smooth functions. Let α and β be regular values of h and g respectively, and let Z_α and W_β be the corresponding level sets. Suppose that, for all points $x \in Z_\alpha \cap W_\beta$, we have:

$$\dim(T_x Z_\alpha \cap T_x W_\beta) = n - 2$$

(a) Show that $Z_\alpha \cap W_\beta$ is an $(n-2)$-dimensional submanifold of Z.

Now generalize this result by:

(b) Replacing h by a smooth function $h : X \to \mathbb{R}^m$ and g by a smooth function $g : X \to \mathbb{R}^k$.

(c) Replacing Z_α and W_β by arbitrary submanifolds of X. *Hint: the question is local!*

5. Let $Z \subset \mathbb{R}^n$ be a 1-dimensional submanifold.

(a) Explain why looking at the tangent space to points in z defines a function:

$$F : Z \to \mathbb{RP}^{n-1}$$

(b) Prove that this function F is smooth. *Hint: pick a chart on Z and consider the inclusion $\iota : Z \hookrightarrow \mathbb{R}^n$.*

(c) Prove that the subset $Z_2 \subset \mathbb{R}^2$ described at the start of Section 3.1 is not a submanifold.

M4P52 Manifolds, 2016
Problem Sheet 6

For questions 1 and 2 we use Definition 5.17 of a 'tangent vector'. We let X be an n-dimensional manifold, and let $\mathcal{A}_x$ denote the set of all charts containing a fixed point $x \in X$.

1. Let $h : X \to \mathbb{R}$ be a smooth function. For any chart $(U, f) \in \mathcal{A}_x$, we can view the Jacobian matrix

$$D(h \circ f^{-1})|_{f(x)} : \mathbb{R}^n \to \mathbb{R}$$

as a vector in $\mathbb{R}^n$. This defines a function from $\mathcal{A}_x$ to $\mathbb{R}^n$. Is it a tangent vector?

Now let Y be a second manifold, of dimension k, and for $y \in Y$ let $\mathcal{B}_y$ denote the set of all charts containing y.

2. Let $F : X \to Y$ be a smooth function, and set $y = F(x)$. Let $\delta : \mathcal{A}_x \to \mathbb{R}^n$ be a tangent vector to x.

   (a) Fix a chart $(U, f) \in \mathcal{A}_x$ and let $\delta_f \in \mathbb{R}^n$ be the value of δ in this chart. Show that the function

$$\begin{aligned} DF|_x(\delta) : \quad & \mathcal{B}_y \to \mathbb{R}^k \\ & (V, g) \mapsto D(g \circ F \circ f^{-1})|_{f(x)}(\delta_f) \end{aligned}$$

   is a tangent vector to y.

   (b) Show that $DF|_x(\delta)$ does not depend on our choice of chart (U, f).

   (c) Show that this construction agrees with our earlier definition of the derivative $DF|_x$.

(Continued on next page.)

⋆3⋆. (a) Let ξ be the vector field on S^1 defined by

$$\xi|_{(x,y)} = (-y, x)^\top \in T_{(x,y)}S^1 \ \subset \mathbb{R}^2$$

(from Example 6.2), and let

$$f_1 : (x, y) \mapsto \tilde{x} = \frac{x}{1+y}$$

be the stereographic projection co-ordinates from Example 2.4. Find the function

$$\widetilde{\xi}_1 : \mathbb{R} \to \mathbb{R}$$

which is the expression for ξ in this chart. *Hint: f_1 can be extended to a function on an open set in $\mathbb{R}^2$. Also note the identity:*

$$1 + \tilde{x}^2 = 2/(1 + y)$$

(b) Now find the expression for ξ in the co-ordinates $f_2 : (x, y) \mapsto \frac{x}{1-y}$ using the transformation law for vector fields.

4. For any $s \in \mathbb{R}$, consider the linear map:

$$\widehat{G}_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$$

(a) Prove that $\widehat{G}_s$ induces a diffeomorphism $G_s : S^2 \to S^2$, and show that this defines a flow G on S^2.

(b) Find the associated vector field ξ^G, and find the points where ξ^G is zero.

5. (a) Show that a vector field on T^2 is the same thing as a smooth function $\widehat{\xi} : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$\widehat{\xi}(x + n, y + m) = \widehat{\xi}(x, y)$$

for all $n, m \in \mathbb{Z}$ and all points $(x, y) \in \mathbb{R}^2$.

(b) Suppose $\widehat{\xi}$ is the constant function

$$\widehat{\xi} : (x, y) \mapsto (u, v)$$

for some fixed $(u, v) \in \mathbb{R}^2$. Find a flow G on T^2 such that ξ^G is the vector field corresponding to $\widehat{\xi}$.

M4P52 Manifolds, 2016
Problem Sheet 7

⋆1⋆. (a) Let $F : X \to Y$ be a smooth function between two manifolds. Fix $x \in X$ and let $y = F(x)$. Show that the linear map

$$\begin{aligned} C^\infty(Y) &\to C^\infty(X) \\ h &\mapsto h \circ F \end{aligned}$$

induces a well-defined map from $T^\star_y Y$ to $T^\star_x X$, and that the rank of this map equals the rank of F at x.

(b) Let Z be a submanifold of $\mathbb{R}^n$. Deduce that for any $x \in Z$ there is a natural surjection $\mathbb{R}^n \to T^\star_x Z$.

2. (a) Let $Z \subset \mathbb{R}^n$ be the level set of a function $h \in C^\infty(\mathbb{R}^n)$ at a regular value, and fix a point $x \in Z$. Show that we can identify $T^\star_x Z$ with the quotient of $\mathbb{R}^n$ by the subspace spanned by the vector $Dh|_x^\top \in \mathbb{R}^n$.

(b) Use this to get an explicit description of the cotangent spaces to S^n.

(c) Generalize part (a) by replacing h with a smooth function $h : \mathbb{R}^n \to \mathbb{R}^k$.

(d) Now suppose $Z \subset X$ is the level set of a smooth function $H : X \to Y$ at a regular value. What can you say about the cotangent spaces to points in Z?

3. Let σ be the curve through $(1, 0, 0) \in S^2$ defined by

$$\begin{aligned} \sigma :&(-\epsilon, \epsilon) \to S^2 \\ &t \mapsto \left(\cos t, \tfrac{1}{\sqrt{2}} \sin t, \tfrac{1}{\sqrt{2}} \sin t\right) \end{aligned}$$

and let $\partial_\sigma \in \mathrm{Der}_{(1,0,0)}(S^2)$ be the corresponding derivation at $(1, 0, 0)$.

(a) Consider the chart with domain $U = S^2 \cap \{x > 0\}$ and co-ordinates:

$$f : (x, y, z) \mapsto (y, z)$$

Write down explicitly the partial derivative operator in $\mathrm{Der}_{(0,0)}(\tilde{U})$ that corresponds to ∂_σ.

(b) Let $h \in C^\infty(S^2)$ be the function:

$$h : (x, y, z) \mapsto \frac{\tan^{-1}(\sinh^{-1} x)}{\log(\cosh(x) + 2)} + xy^3 + x^2 z$$

Compute $\partial_\sigma h$. *Hint: think before you calculate.*

Continued on next page.

4. In this exercise we'll prove Proposition 7.8 'in reverse'. Fix a point x in a manifold X.

   (a) For any $h \in C^\infty(X)$, define a function:

   $$\begin{aligned} T_x X &\to \mathbb{R} \\ [\sigma] &\mapsto \partial_\sigma(h) \end{aligned}$$

   Show that this function is well-defined and linear.

   (b) Show that the resulting function

   $$C^\infty(X) \to (T_x X)^\star$$

   is linear, and induces a well-defined injection $T_x^\star X \to (T_x X)^\star$.

   (c) Prove that this is actually an isomorphism $T_x^\star X \xrightarrow{\sim} (T_x X)^\star$. *You may use other results from the course at this point.*

   (d) Convince yourself that this is the dual to the isomorphism in Proposition 7.8.

5. Let $x \in X$ be a point in a manifold. Let $\mathfrak{d} : C^\infty(X) \to \mathbb{R}$ be a linear map which vanishes on the subspace $R_x(X)$. Show that $\mathfrak{d}$ is a derivation at x.

6. Let $F : X \to Y$ be a smooth function, fix $x \in X$, and let $y = F(x)$. Let

   $$DF|_x : \mathrm{Der}_x(X) \to \mathrm{Der}_y(Y)$$

   be the dual linear map to the map defined in question 1(a).

   (a) If σ is a curve through x, what does the operator $DF|_x(\partial_\sigma)$ do to a function $h \in C^\infty(Y)$? Show that $DF|_x$ agrees with our previous definitions of the derivative.

   (b) Using this definition, prove that the chain rule holds.

M4P52 Manifolds, 2016
Problem Sheet 8

$\star$1$\star$. (a) Let X be the open ball $B_1\big((0,0)\big) \subset \mathbb{R}^2$, and define a vector field $\tilde{\xi}$ on X by:

$$\tilde{\xi} : (x, y) \mapsto (0, \sqrt{1 - x^2 - y^2})$$

Viewing $\tilde{\xi}$ as operator in $\mathrm{Der}(X)$, evaluate it on the function $x^2 + y^2 \in C^\infty(X)$.

(b) Let ξ be the vector field on S^2 defined by:

$$\xi : (x, y, z) \mapsto (0, z, -y)$$

Let $h \in C^\infty(S^2)$ be the function $h : (x, y, z) \mapsto z^2$. Find a function $g \in C^\infty(\mathbb{R}^3)$ such that $\xi(h) = g|_{S^2}$.

(c) What's the connection between parts (a) and (b)? *State the relationship clearly, but you don't need to provide a detailed justification.*

2. (a) Prove that $C^\infty(\mathbb{RP}^{n-1})$ can be identified with the space of all smooth functions

$$h : \mathbb{R}^n \setminus 0 \to \mathbb{R}$$

which obey the condition:

$$h(\lambda x) = h(x), \quad \forall \lambda \in \mathbb{R} \setminus 0,\ x \in \mathbb{R}^n \setminus 0$$

(b) Let $x_1, ..., x_n$ be the standard co-ordinates on $\mathbb{R}^n$. Show that for any $i, j \in [1, n]$ the operator $x_i \frac{\partial}{\partial x_j}$ defines a vector field on $\mathbb{RP}^{n-1}$. Deduce that there is a linear map

$$\mathrm{Mat}_{n\times n}(\mathbb{R}) \longrightarrow \mathrm{Der}(\mathbb{RP}^{n-1})$$

but show that this is not an injection.

(c) *(Advanced)* See how much of this you can generalize to the Grassmannian $\mathrm{Gr}(k, n)$.

3. (a) Let $h \in C^\infty(\mathbb{R}^2)$ be the function $h(x, y) = x^2 y$. Write down the one-form dh.

(b) Let α_+ and α_- be the one-forms

$$\alpha_\pm = \cos y\, dx \pm x \sin y\, dy$$

on $\mathbb{R}^2$. Does there exist a function $h_+ \in C^\infty(\mathbb{R}^2)$ such that $dh_+ = \alpha_+$? Does there exist a function $h_- \in C^\infty(\mathbb{R}^2)$ such that $dh_- = \alpha_-$?

4. In Example 8.5, verify that $d\tilde{h}_1$ transforms into $d\tilde{h}_2$ under the transition function between the two charts.

5. If we have a vector field ξ and a one-form α on a manifold X, show that we can combine them to get a function $g_{\xi,\alpha} \in C^\infty(X)$. If $\alpha = dh$ for some $h \in C^\infty(X)$, find another description of $g_{\xi,\alpha}$.

M4P52 Manifolds, 2016
Problem Sheet 9

1. Consider the chart on S^n with domain $U = S^n \cap \{x_0 > 0\}$ and co-ordinates:

$$f : (x_0, x_1, ..., x_n) \mapsto (x_1, ..., x_n)$$

   (a) Let $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion. For each $i \in [0, n]$, find the expression for the one-form $\iota^* dx_i$ in the chart (U, f).

   (b) Use (a) to find the expression of the one-form $\iota^*(x_0\, dx_0 + ... + x_n\, dx_n)$ in the chart (U, f). Now find another way to get to this answer.

2. Let α be the one-form

$$\alpha = y\, dx - x\, dy$$

on $\mathbb{R}^2$, and let $\iota : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map.

   (a) Show that $\iota^*\alpha$ is not zero at any point.

   (b) Is there a function $h \in C^\infty(S^1)$ such that $\iota^*\alpha = dh$?

3. Convince yourself that, for a general smooth function $F : X \to Y$, it is not possible to 'pull-back' a vector field along F. Now find a hypothesis on F that makes it possible.

4. Let V be a four-dimensional vector space with a basis e_1, e_2, e_3, e_4.

   (a) Write down a basis for $\bigwedge^3 V^\star$.

   (b) If $c \in \bigwedge^2 V^\star$ is decomposable, show that $c \wedge c = 0$. Find an element of $\bigwedge^2 V^\star$ that is not decomposable.

5. Let V be a vector space, let $c \in V^\star$ and let $\hat{c} \in \bigwedge^2 V^\star$. For any three vectors $v_1, v_2, v_3 \in V$, find an expression for the value of

$$(c \wedge \hat{c})(v_1, v_2, v_3) \ \in \mathbb{R}$$

Hint: start by assuming that $\hat{c}$ is decomposable. If you have the energy, try this question again for the case that both c and $\hat{c}$ lie in $\bigwedge^2 V^\star$.

6. Let $h, g \in C^\infty(\mathbb{R}^2)$ be the functions $h(x, y) = x^2 y$ and $g(x, y) = \sin(xy)$. Find the two-form $dh \wedge dg$.

7. In Example 8.30 we saw that the curl operator $\nabla\times$ on vector fields in 3-dimensions can be interpreted as a special case of the exterior derivative d. Find similar interpretations of the gradient ∇ and divergence $\nabla\cdot$ operators. What does Proposition 8.31(i) say in this situation?

8. Let (U, f) be the chart on S^n from Question 1, and let $\tilde{x}_1, ..., \tilde{x}_n$ be the standard co-ordinates on the codomain $\tilde{U}$ of this chart.

    (a) Let $\iota : S^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map. Find the expression of the n-form $\iota^*(dx_0 \wedge dx_1 \wedge ... \wedge dx_{n-1})$ in the chart (U, f).

    (b) For the $(n-1)$-form

    $$\tilde{\alpha} = \left(1 - \sum_{i=1}^{n} \tilde{x}_i^2\right)^{\frac{1}{2}} d\tilde{x}_1 \wedge ... \wedge d\tilde{x}_{n-1} \quad \in \Omega^1(\tilde{U})$$

    compute $d\tilde{\alpha}$. Explain the relationship to your answer for (a).

9. (a) Let V be a vector space. Given $c \in \bigwedge^p V^\star$, and $v \in V$, show that they can be combined to get an element $i_v c \in \bigwedge^{p-1} V^\star$. Now choose a basis for V, and describe $i_v c$ in the case that both v and c are basis elements.

    (b) Let X be a manifold. Deduce that if we are given $\alpha \in \Omega^p(X)$ and ξ a vector field on X, we can combine them to get a $(p-1)$-form $i_\xi \alpha$. Prove that if α and ξ are smooth then $i_\xi \alpha$ is also smooth.

10. (a) Show that a smooth function $F : X \to Y$ induces a linear map

    $$F^\star : H^p_{dR}(Y) \to H^p_{dR}(X)$$

    for any p.

    (b) Show that the wedge product

    $$\wedge : H^p_{dR}(X) \times H^q_{dR}(X) \to H^{p+q}_{dR}(X)$$

    is well-defined, for any p, q.

    (c) What is the topological meaning of the number $\dim H^0_{dR}(X)$? *Hint: what kind of function $h \in C^\infty(X)$ satisfies $dh = 0$?*

M4P52 Manifolds, 2016
Mastery Material Problem Sheet

1. Let x, y, z be the standard co-ordinates on $\mathbb{R}^3$. Let Z be the level set at a regular value of a function $h \in C^\infty(\mathbb{R}^3)$, and let $\iota : Z \hookrightarrow \mathbb{R}^3$ denote the inclusion. Let ω be the two-form:

$$\omega = dx \wedge dy \quad \in \Omega^2(\mathbb{R}^3)$$

   Show that if $\frac{\partial h}{\partial z} \neq 0$ at all points in Z then $\iota^*\omega$ is a volume form on Z.

2. Let X be a compact oriented n-dimensional manifold. For any $\alpha \in \Omega^p(X)$ and $\beta \in \Omega^{n-p-1}(X)$ show that:

$$\int_X d\alpha \wedge \beta = \pm \int_X \alpha \wedge d\beta$$

3. For any manifold X, let $\mathcal{O}r(X)$ denote the set of all possible orientations on X. Let X and Y be two n-dimensional manifolds.

   (a) Let $F : X \to Y$ be a smooth function which has rank n at all points. If ω is a volume form on Y, show that $F^\star\omega$ is a volume form on X. Show that the function

$$\begin{aligned} F^\star : \mathcal{O}r(Y) &\to \mathcal{O}r(X) \\ [\omega] &\mapsto [F^\star\omega] \end{aligned}$$

   is well-defined.

   (b) Suppose that that X is connected and orientable. Show that $\mathcal{O}r(X)$ contains exactly two elements.
   Deduce that a diffeomorphism $G : X \to X$ induces a bijection from $\mathcal{O}r(X)$ to $\mathcal{O}r(X)$ which is either the identity, or a transposition.

   In the first case we say that G is *orientation-preserving*, and in the second case we say that G is *orientation-reversing.*

   (c) Suppose X is connected and orientable, and let $G : X \to X$ be an orientation-reversing diffeomorphism. Now suppose that $q : X \to Y$ is a smooth function having rank n at all points, satisfying:

$$q \circ G = q$$

   Prove that Y cannot be orientable.

(d) Show that the Klein bottle K (Sheet 1, Q5) is not orientable.

(e) Show that the function

$$G : x \mapsto -x$$

is a diffeomorphism from S^n to itself.

Now let $\omega_0 = dx_1 \wedge ... \wedge dx_{n+1}$ be the standard volume form on $\mathbb{R}^{n+1}$, and let ω' be the induced volume form on the submanifold S^n (as in Proposition 9.6). Fix the point $p = (0, ..., 0, 1) \in S^n$, and consider the linear map:

$$\wedge^n (DG|_p)^\star : \wedge^n T^\star_{-p} S^n \to \wedge^n T^\star_p S^n$$

Show that applying this map to the element $\omega'|_{-p}$ produces either ω'_p or $-\omega'_p$, depending on whether n is odd or even. *Hint: consider $T_{\pm p} S^n$ as subspaces of* $\mathbb{R}^{n+1}$.

Deduce that $G : S^n \to S^n$ is orientation-preserving iff n is odd.

(f) Show that $\mathbb{RP}^n$ is not orientable if n is even.

4. (a) Prove that a 2-form α on the torus T^2 is the same thing as 2-form on $\mathbb{R}^2$

$$\hat{\alpha} = \hat{h}(x, y)\, dx \wedge dy \quad \in \Omega^2(\mathbb{R}^2)$$

satisfying $\hat{h}(x + n, y + m) = \hat{h}(x, y)$ for all $n, m \in \mathbb{Z}$.

(b) In Example 2.12 we found an atlas on the torus T^2 with four charts $(U_i, f_i), 1 \leq i \leq 4$. Find a volume form ω on T^2 such that each of these charts is oriented with respect to ω.

(c) Show that there is a partition-of-unity on T^2 consisting of four functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ such that each φ_i is only non-zero inside the chart U_i. *Hint: it can be constructed from the partition-of-unity on T^1 found in Example 9.17.*

(d) Pick $\alpha \in \Omega^2(T^2)$, and let $\hat{\alpha} = \hat{h}\, dx \wedge dy$ be the corresponding periodic two-form on $\mathbb{R}^2$ as in part (a). Show that

$$\int_{T^2} \alpha = \int_{y=0}^{1} \int_{x=0}^{1} \hat{h}(x, y)\, dx\, dy$$

(using the orientation $[\omega]$ as in part (b)). Now prove that there is no $\beta \in \Omega^1(T^2)$ such that $\omega = d\beta$.

5. (a) Let X be a manifold, and let $Z \subset X$ be a k-dimensional submanifold which is compact and oriented. Use Z to construct a linear map from $H^k_{dR}(X)$ to $\mathbb{R}$.

(b) Let α and β be the closed one-forms on T^2 corresponding to the periodic one-forms dx and dz on $\mathbb{R}^2$ (see e.g. Q4, part (a)). Construct two linear maps $a, b \in (H^1_{dR}(T^2))^*$ such that

$$a([\alpha]) \neq 0, \quad a([\beta]) = 0, \quad b([\alpha]) = 0, \quad b([\beta]) \neq 0$$

and deduce that $H^1_{dR}(T^2)$ is at least two-dimensional.

M4P52 Manifolds, 2016
Vector Bundles Problem Sheet

1. Let $F : X \to Y$ be a smooth function between two manifolds, and define a function
$$DF : TX \to TY$$
between their tangent bundles by:
$$DF : (x, v) \to \big(F(x), DF|_x(v)\big)$$
Show that DF is smooth.

2. Let $\pi : E \to X$ be a vector bundle, let $\xi : X \to E$ be a section, and let
$$\Gamma_\xi = \{(x, \xi|_x),\ x \in X\} \subset E$$
denote the graph of ξ. Show that Γ_ξ is a submanifold of E, and that Γ_ξ is diffeomorphic to X.

3. Let $\pi : E \to T^1$ be the 'infinite Möbius strip' vector bundle from Example E.5. Show that a section of E is the same thing as a smooth function $\hat{\sigma} : \mathbb{R} \to \mathbb{R}$ satisfying
$$\hat{\sigma}(x + 1) = -\hat{\sigma}(x)$$
for all $x \in \mathbb{R}$. Prove that E is not trivial.

4. Prove that T^n is parallelizable for any n.

5. Let X be a parallelizable manifold.
   (a) Prove that $T^\star X$ is trivial. *(Hint: dual bases.)*
   (b) Now prove that $\wedge^p T^\star X$ is trivial, for all p.